(1) (a) $\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0$ , $\theta(x,y)=X(x)Y(y)$

$\frac{d^2X}{dx^2}Y+X\frac{d^2Y}{dy^2}=0$ $\quad \frac{1}{X}\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}=\lambda$ (const)

I) $\lambda=-p^2<0$ $\quad \frac{d^2X}{dx^2}+p^2X=0$, $\frac{d^2Y}{dy^2}-p^2Y=0$ $\quad \begin{cases} X(x)=A\cos px+B\sin px \\ Y(y)=C\cosh py+D\sinh py\end{cases}$

II) $\lambda=0$ $\quad \frac{d^2X}{dx^2}=0$, $\frac{d^2Y}{dy^2}=0$ $\quad \begin{cases} X(x)=Ax+B \\ Y(y)=Cy+D\end{cases}$

III) $\lambda=p^2>0$ $\quad \frac{d^2X}{dx^2}-p^2X=0$, $\frac{d^2Y}{dy^2}+p^2Y=0$ $\quad \begin{cases} X(x)=A\cosh px+B\sinh px \\ Y(y)=C\cos py+D\sin py\end{cases}$

General solution has $\theta=$ I + II + III with $A,B,C,D,p$ arbitrary.

(b) $\theta(x,0)=\theta(x,1)=0$ $\Rightarrow$ Type III solution $Y(y)=D_n\sin n\pi y$

$(p=n\pi,\ n=1,2,3\ldots)$

$X(x)=A_n\cosh n\pi x+B_n\sinh n\pi x$ $\qquad \theta(0,y)=0 \Rightarrow A_n=0\ \forall n$

Solution is $\theta(x,y)=\sum_{n=1}^{\infty}A_n\sinh n\pi x\ \sin n\pi y$

$\theta(1,y)=1$ $\qquad \sum_{n=1}^{\infty}A_n\sinh n\pi\cdot\sin n\pi y=1$

Use Fourier series $\quad \frac{1}{2}A_m\sinh m\pi=\int_0^1\sin m\pi y\,dy$

$=\left[-\frac{1}{m\pi}\cos m\pi y\right]_0^1$

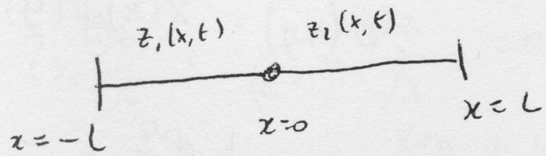
$=\frac{1}{m\pi}\left(1-(-1)^m\right)=\begin{cases}\frac{2}{m\pi} & m \text{ odd}\\ 0 & m \text{ even}\end{cases}$

$A_m=\begin{cases}\frac{4}{m\pi\sinh m\pi} & m \text{ odd}\\ 0 & m\text{ even}\end{cases}$ , use $n=2j+1$, $j=0,1,2\ldots$

$\theta(x,y)=\sum_{j=0}^{\infty}\frac{4}{(2j+1)\pi\sinh(2j+1)\pi}\sinh(2j+1)\pi x\cdot\sin(2j+1)\pi y$ (*)

(c) Symmetry, linearity $\Rightarrow$ $\theta(x,y)=(*)+\sum_{j=0}^{\infty}\frac{4}{(2j+1)\pi\sinh(2j+1)\pi}\sin(2j+1)\pi x\cdot\sinh(2j+1)\pi(1-y)$

②




A mass $M$ is attached to the mid-point of a string length $2L$ density $\rho$, under tension $T$ as shown. Find the periods of the normal modes.

<u>Equations</u> $\quad \frac{1}{c^2}\frac{\partial^2 z_1}{\partial t^2} = \frac{\partial^2 z_1}{\partial x^2} \quad (-L \le x \le 0)$ , $\quad \frac{1}{c^2}\frac{\partial^2 z_2}{\partial t^2} = \frac{\partial^2 z}{\partial x^2} \quad (0 \le x \le L)$ $\qquad c = \sqrt{T/\rho}$

<u>B.C.s</u>

① $z_1(-L,t) = 0$

② $z_2(L,t) = 0$

③ $z_1(0,t) = z_2(0,t)$

$z_{1t}(0,t) = z_{2t}(0,t)$

④ $M\frac{\partial^2 z_1}{\partial t^2}(0,t) = T\left(\frac{\partial z_2}{\partial x} - \frac{\partial z_1}{\partial x}\right)(0,t)$

Use separation of variables in each half of the string

$$z_1(x,t) = \left[A\cos p(x+L) + B\sin p(x+L)\right]e^{i\omega t} \qquad A, B \text{ complex}$$

$$z_2(x,t) = \left[C\cos p(x-L) + D\sin p(x-L)\right]e^{i\omega t} \qquad C, D \quad ''$$

Equations give $\quad -p^2 = \frac{-\omega^2}{c^2} \qquad \omega = cp$

B.C. ① $\Rightarrow$ $A=0$ $\qquad z_1 = B\sin\frac{\omega}{c}(x+L)\,e^{i\omega t}$

B.C. ② $\Rightarrow$ $C=0$ $\qquad z_2 = D\sin\left(\left(\frac{\omega}{c}\right)(x-L)\right)e^{i\omega t}$

③ <u>At $x=0$</u> $\quad z_1 = z_2 \qquad B\sin\frac{\omega}{c}L = -D\sin\frac{\omega L}{c}$

Either Ⓘ $B=-D$ $\quad$ "Symmetric modes"

or Ⓘⓘ $\frac{\omega L}{c} = n\pi$

④ <u>At $x=0$</u> $\quad -M\omega^2 B\sin\frac{\omega L}{c}\,e^{i\omega t} = \frac{T\omega}{c}\left(D\cos\frac{\omega L}{c} - B\cos\frac{\omega L}{c}\right)e^{i\omega t}$

Ⓘ $B=-D$ $\qquad BM\omega^2\sin\frac{\omega L}{c} = \frac{2T\omega}{c}\cos\frac{\omega L}{c}$ , $\quad c^2 = T/\rho$

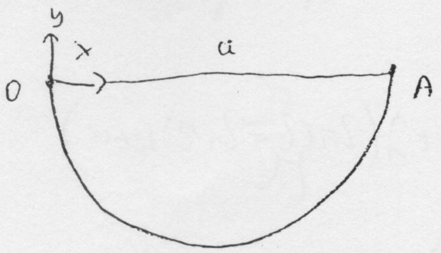
$$\left(\frac{\omega L}{c}\right)\tan\left(\frac{\omega L}{c}\right) = \frac{2\rho L}{M}. \qquad \text{As } M\to\infty \quad \tan\frac{\omega L}{c}\to\infty$$

$$\frac{\omega L}{c} = \frac{(2j+1)\pi}{2} \qquad j = 0, 1, 2, 3 \ldots$$

Ⓘⓘ <u>Anti-symmetric modes</u> $\quad$ (④ gives <u>$B=D$</u>) $\qquad \frac{\omega L}{c} = n\pi \qquad n = 1, 2, 3, \ldots$

(3)

Define the origin at one end of the chain as shown




$$V = -\rho g \int_0^A y\,ds = -\rho g \int_0^a y(1+y'^2)^{1/2}\,dx$$

$$L = \int_0^A ds = \int_0^a (1+y'^2)^{1/2}\,dx = l \quad (\text{cons.})$$

Need to minimize $V$ with $L$ constant.
Find extremal function of $V + \lambda L$,

$$V + \lambda L = \int_0^a H(y, y')\,dx$$

with $H(y,y') = -\rho g\, y(1+y'^2)^{1/2} + \lambda (1+y'^2)^{1/2}$

No explicit dependence on $x$ - use Beltrami

$$y' \frac{\partial H}{\partial y'} - H = A \quad (\text{cons.})$$

$$(\rho g y - \lambda)\left[1+y'^2\right]^{-1/2} = A$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{A^2}\left[(\rho g y - \lambda)^2 - A^2\right] = k^2(y+h)^2 - 1 \qquad \lambda = -\rho g h,\quad k = \frac{\rho g}{A}$$

$$\pm \int \frac{dy}{\left((y+h)^2 + \frac{1}{k^2}\right)^{1/2}} = kx + \alpha, \quad \alpha \text{ - cons.}$$

Substitute $y + h = \frac{1}{k}\cosh\theta$ and integrate to give

$$\pm \cosh^{-1} k(h+y) = kx + \alpha$$

$$y = -h + \frac{1}{k}\cosh(kx+\alpha) \qquad (\text{choose +ve to give Min. P.E.})$$

$y(0) = 0$, $y(a) = 0$ gives

$kh = \cosh\alpha = \cosh(ka+\alpha)$
$\Rightarrow \alpha = -\frac{ka}{2}$.

(A) $kh = \cosh\frac{ka}{2}$. Final equation is obtained from $L$:

$$\int_0^a [1+y'^2]^{1/2}\,dx = \int_0^a \left[1 + k^2(y+h)^2 \ \ 1\right]^{1/2} dx = \int_0^a \cosh(kx+\alpha)\,dx$$

$\alpha = \frac{ka}{\ }$, (B) $2\sinh\frac{ka}{2} = kl$

Equations (A) and (B) determine $k$ and $h$.

4

$$V = \pi R^2 L + \tfrac{2}{3}\pi R^3 \tan\alpha$$

$$S = 2\pi R L + 2\pi R^2 \sec\alpha \quad = \text{cons.}$$

Find critical points of

$$H(R, L, \alpha) = V + \lambda S = \left(\pi R^2 L + \tfrac{2}{3}\pi R^3 \tan\alpha\right) + \lambda\left(2\pi R L + 2\pi R^2 \sec\alpha\right)$$

(1) $$\frac{\partial H}{\partial R} = 2\pi R L + 2\pi R^2 \tan\alpha + \lambda\left(2\pi L + 4\pi R \sec\alpha\right) = 0$$

(2) $$\frac{\partial H}{\partial L} = 2\pi R \lambda + \pi R^2 = 0$$

(3) $$\frac{\partial H}{\partial \alpha} = 2\pi R^2 \lambda \sec\alpha \tan\alpha + \tfrac{2}{3}\pi R^3 \sec^2\alpha = 0$$

(2) gives $\lambda = -R/2$

(3) gives $$\pi R^3\left(-\sec\alpha\tan\alpha + \tfrac{2}{3}\sec^2\alpha\right) = 0$$

$$\sin\alpha = \tfrac{2}{3}, \quad \cos\alpha = \frac{\sqrt{5}}{3}, \quad \tan\alpha = \tfrac{2}{\sqrt{5}}$$

(1) gives $$\pi R L + 2\pi R^2(\tan\alpha - \sec\alpha) = 0$$

$$\pi R L = \frac{2\pi R^2}{\sqrt{5}} \qquad L = \tfrac{2}{\sqrt{5}} R$$

$$S = \pi R^2\left(\frac{4}{\sqrt{5}} + \frac{6}{\sqrt{5}}\right) = 2\sqrt{5}\,\pi R^2$$

$$R = \left(\frac{S}{2\sqrt{5}\,\pi}\right)^{1/2}, \quad L = \frac{2}{\sqrt{5}}\left(\frac{S}{2\sqrt{5}\,\pi}\right)^{1/2}, \quad \alpha = \tan^{-1}\left(\tfrac{2}{\sqrt{5}}\right)$$

Maximum volume $V$ is $$V = \pi\left(\frac{2}{\sqrt{5}} + \frac{2}{3}\cdot\frac{2}{\sqrt{5}}\right)\left(\frac{S}{2\sqrt{5}\,\pi}\right)^{3/2}$$

$$= \frac{2\sqrt{5}}{3}\pi\left(\frac{S}{2\sqrt{5}\,\pi}\right)^{3/2} = \frac{1}{3}\frac{S^{3/2}}{(2\sqrt{5}\,\pi)^{1/2}}$$

**Q5.**

(a)

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{1}{x^2 - y^2}$$

Use change of variables $\xi = x$, $\eta = x + y$

$$\frac{\partial z}{\partial \xi} = \frac{1}{\xi^2 - (\xi - \eta)^2} = \frac{1}{2\xi\eta - \eta^2}$$

Integrate $\quad z = f(\eta) + \frac{1}{2\eta} \ln(2\xi\eta - \eta^2)$

$$= f(x+y) + \frac{1}{2(x+y)} \ln|x^2 - y^2| \qquad f \text{, arbitrary.}$$

(b)

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = \sin(x + 3y)$$

Find C.F. $\quad$ Aux. eqn. $\quad m^2 - m - 2 = 0 \qquad m = 2, -1 \qquad \left[z = f(y + mx)\right]$

$$z_{cf} = f(y - x) + g(y + 2x)$$

Find P.I. $\quad$ Try $\quad z = A\sin(x + 3y)$

$$z_{xx} = -A\sin(x+3y)$$
$$z_{xy} = -3A\sin(x+3y)$$
$$z_{yy} = -9A\sin(x+3y)$$

$$-A + 3A + 18A = 1$$

$$A = 1/20$$

$$z = f(y-x) + g(y+2x) + 1/20 \sin(x+3y) \qquad (f, g \text{ arbitrary}).$$

(c)

$$xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = x^2 - y^2$$

Assoc. ODES $\qquad \overset{①}{\frac{dx}{xz}} = \overset{②}{\frac{dy}{yz}} = \overset{③}{\frac{dz}{x^2 - y^2}}$

①+② $\qquad \frac{dy}{dx} = \frac{y}{x} \qquad \int \frac{dy}{y} = \int \frac{dx}{x} \qquad \ln y = \ln x + \ln c$

$$c = y/x$$

①+③ $\qquad \frac{dz}{x^2(1-c^2)} = \frac{dx}{xz} \qquad \int z\,dz = \int x(1-c^2)\,dx$

$$\tfrac{1}{2}z^2 = \tfrac{1}{2}x^2(1-c^2) + \tfrac{1}{2}d \qquad d = z^2 - x^2 + y^2$$

General sltn in implicit form is $\quad F\left(\frac{y}{x}, z^2 - x^2 + y^2\right) = 0 \qquad (F \text{ arbitrary})$

6 (a) $\quad 2x\dfrac{\partial z}{\partial t} + \dfrac{1}{(t+1)^2}\dfrac{\partial z}{\partial x} = 0 \quad , \quad z(x,0) = \begin{cases} 1-x^2 & \\ 0 & |x|> \end{cases}$



Assoc. ODE $\quad \dfrac{dx}{dt} = \dfrac{1}{2x(t+1)^2} \qquad \int 2x\,dx = \int \dfrac{dt}{(t+1)^2}$

$$x^2 + \frac{1}{t+1} = \eta \quad \text{cons.}$$

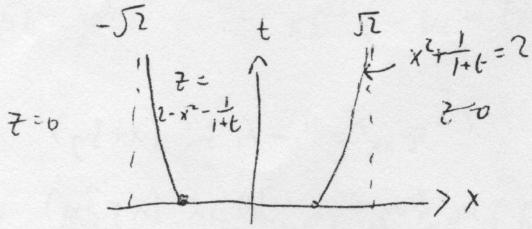
Use C.O.V. with $\xi = t$ , $\eta = x^2 + 1/_{t+1}$

$$\frac{\partial z}{\partial \xi} = 0 \quad , \quad z = f\left(x^2 + \tfrac{1}{t+1}\right)$$

Need to determine arbitrary function $f$ : At $t=0$ $\quad f(x^2+1) = \begin{cases} 1-x^2 & x^2 \le 1 \\ 0 & x^2 > 1 \end{cases}$

write $s = 1+x^2 \qquad f(s) = \begin{cases} 2-s & s \le 2 \\ 0 & s > 2 \end{cases}$

So solution is $\quad z(x,t) = \begin{cases} 2 - x^2 - \dfrac{1}{1+t} , & x^2 + \dfrac{1}{t+1} \le 2 \\ 0 & x^2 + \dfrac{1}{t+1} > 2 \end{cases}$




(b) $\quad \dfrac{\partial^2 z}{\partial t^2} = \dfrac{\partial^2 z}{\partial x^2} \qquad z(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x \ge 0 \end{cases} \qquad z_t(x,0) = 0.$

D'Alembert gives $\quad z(x,t) = \tfrac{1}{2}\left[F(x+t) + F(x-t)\right] \qquad F(x) = \begin{cases} 1 & x < 0 \\ 0 & x \ge 0 \end{cases}$

$$z(x,t) = \tfrac{1}{2}\begin{cases} 1 & x+t < 0 \\ 0 & x+t \ge 0 \end{cases} + \tfrac{1}{2}\begin{cases} 1 & x-t < 0 \\ 0 & x-t \ge 0 \end{cases}$$

$$= \begin{cases} 1 & x < -t & \text{(I)} \\ \tfrac{1}{2} & -t \le x < t & \text{(II)} \\ 0 & x \ge t & \text{(III)} \end{cases}$$